Branched Coverings, Degenerations, and Related Topics
2009 Mar. 9

Dehn surgery along A'Campo's divide knots, Lens spaces and plane curves

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0. Short survey (intro)
1. Dehn surgery
2. A'Campo's divide knots
3. Lens space surgery
4. Results (old and new)

Lens space surgery is related to plane curves of special type

§0. Short survey of this talk

Lens space $L(p, q)$ is a "simple" 3-manifold, the quotient space of S^3 ($\subset \mathbf{C}^2$) by the cyclic action of $\mathbf{Z}/p\mathbf{Z}$:

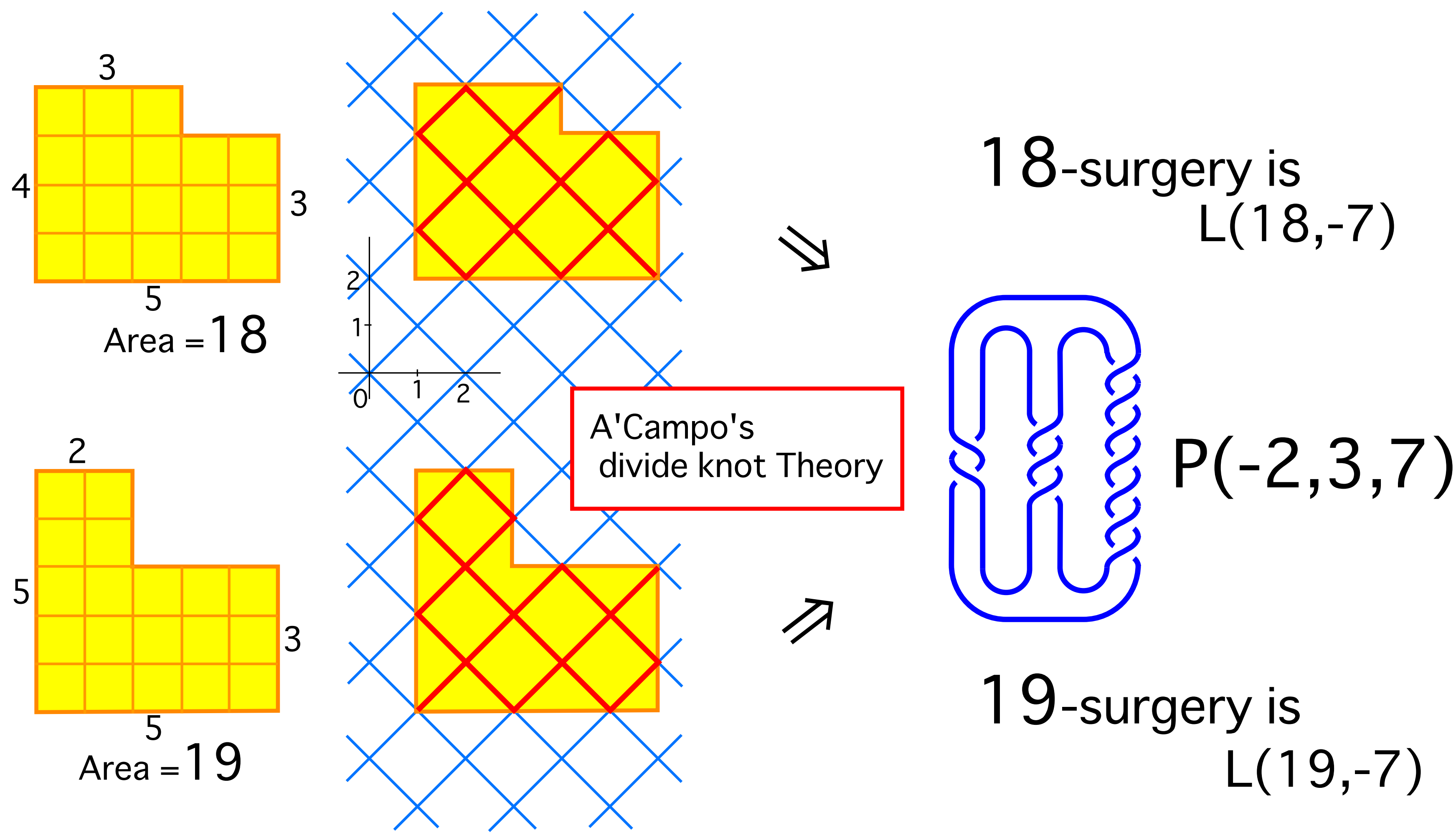
$$\zeta \circ (z, w) = (\zeta z, \zeta^q w),$$

where $\zeta = \exp(2\pi\sqrt{-1}/p)$.
$L(p, q)$ is obtained by gluing two solid tori alng the boundaries.

Lens space surgery is a researching area in low-dimensional topology, asking why (how)

such a simple manifold can be obtained from complicated knots

by surgery *unexpectedly*. It is $30 \sim 40$ years old.



Study lens space surgery by plane curves.

My purpose is to show

Lens space surgery is related to plane curves of special type

By another approach (Heegarrd Floer homology, **C**-links by Rudolph $\cdots$), it is proved:

Lemma. [Hedden]
Any knots of lens space surgery is intersection of an algebraic surface in $\mathbf{C}^2$ *and* B^4. ■

My study is more concrete, to know
· the construction of each knot of lens space surgery,
· the combinaiton of blow-downs,
· the set of lens space surgeries. $\cdots$

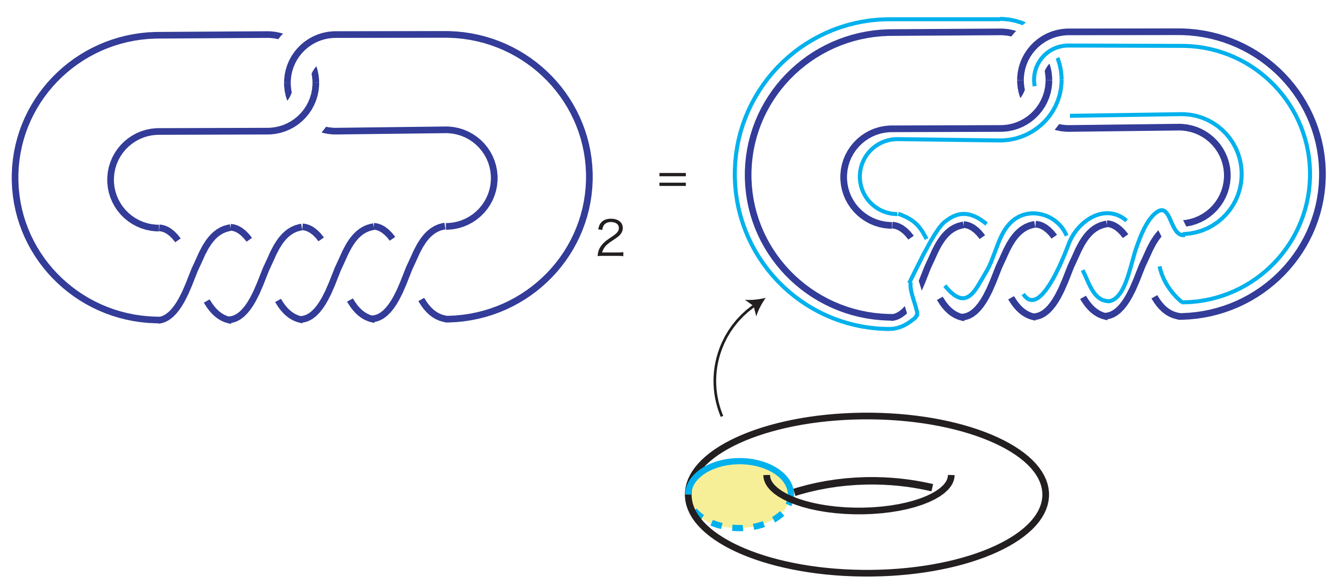
§1. Dehn surgery

Dehn surgery = Cut and paste of a soliod torus.

$$(K;p) := \left(S^3 \setminus \text{open nbd} N(K)\right) \cup_\partial \text{Solid torus.}$$

Coefficient (in $\mathbf{Z}$) "framing" = a parallel *curve* of K,
or the linking number.

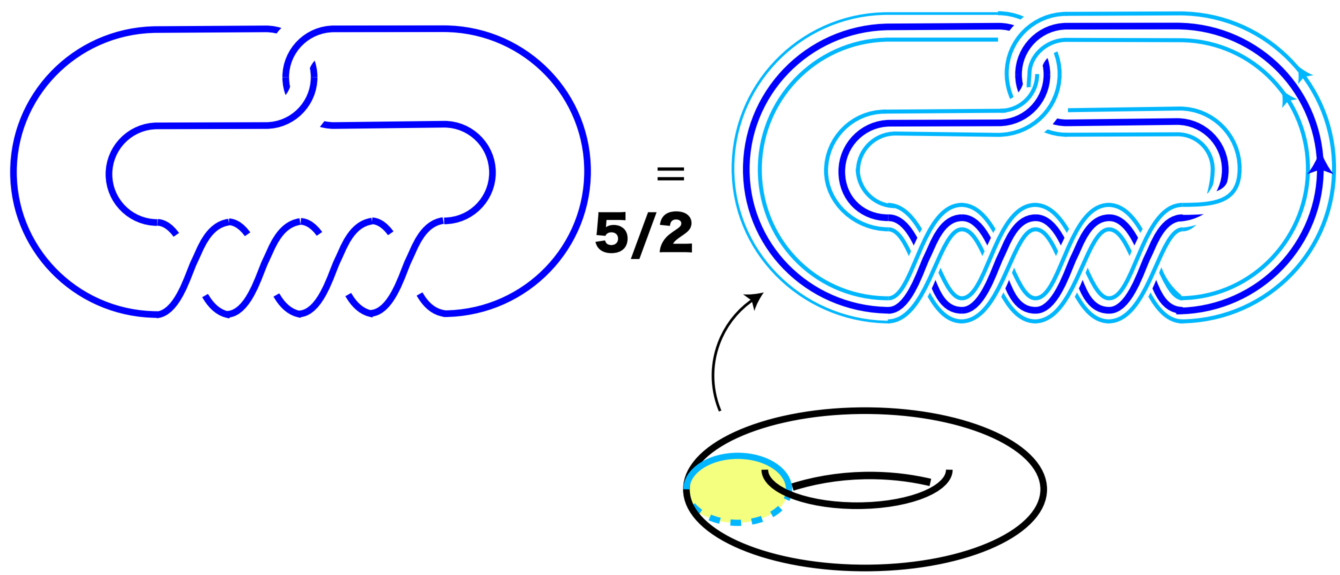
Solit torus is reglued such as "the meridian comes to the parallel"



Coefficient can be p/q in $\mathbf{Q}$: the meridian comes to

$$p[m_K] + q[l_K] \quad \text{in } H_1(\partial N(K)) = \mathbf{Z} \oplus \mathbf{Z}$$

where m_K (and l_K) is the meridian, (prefered longitude) of K.



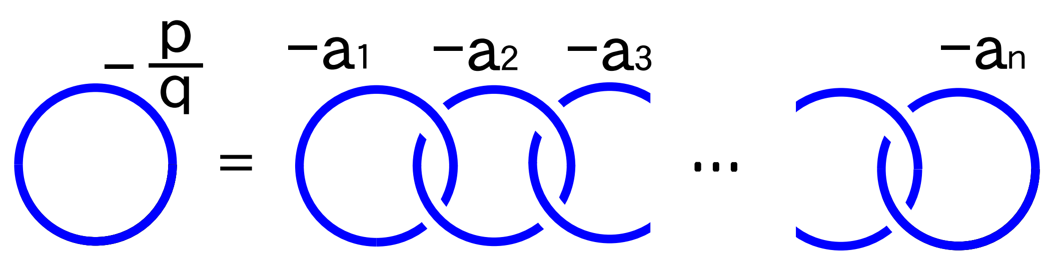
Thm. [Lickorish '62]

Any closed connected oriented 3-manifold M is obtained by a framed link (L, p) in S^3, i.e., $M = (L; p)$,

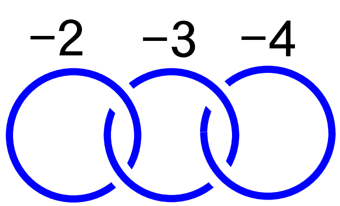
$$(L, p) \ = \ (K_1, p_1) \cup (K_2, p_2) \cup \cdots \cup (K_n, p_n).$$

Framed Links for Lens space $L(p,q)$

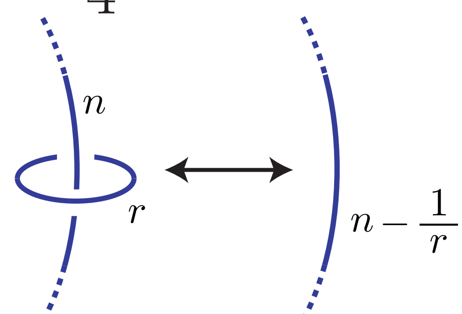
$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n}}} \qquad (a_i > 1)$$



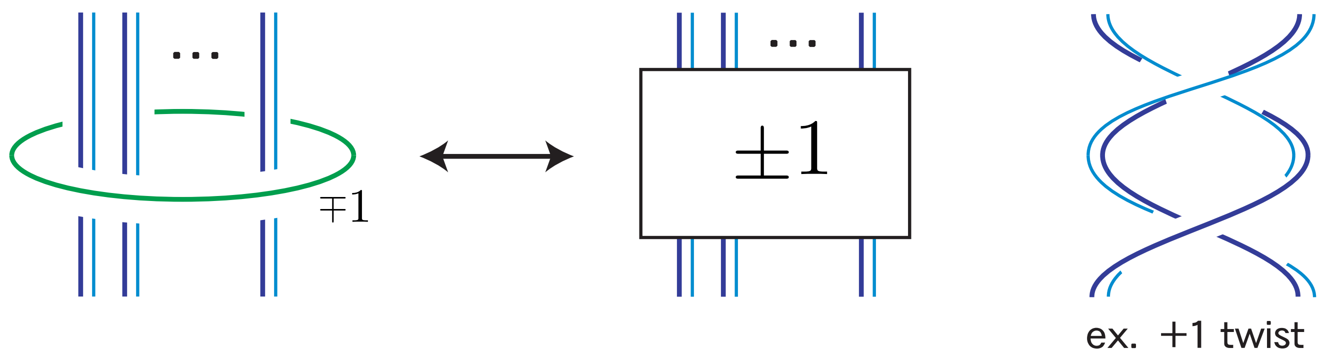
$L(18,11)$ $(= L(18,5))$ $\quad \frac{18}{11} = 2 - \cfrac{1}{3 - \cfrac{1}{4}}, \quad \frac{18}{5} = 4 - \cfrac{1}{3 - \cfrac{1}{2}}$



For $n \in \mathbf{Z}, r \in \mathbf{Q}$

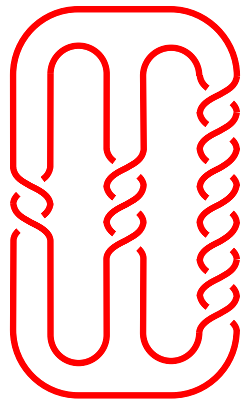


Thm. Kirby calclus ([Fenn-Rourke] ver.)
The 3-manifolds are homeo. $(L;p) \cong (L';p')$
$\Leftrightarrow$ framed links $(L,p), (L',p')$ are moved to each other as below and isotopy:



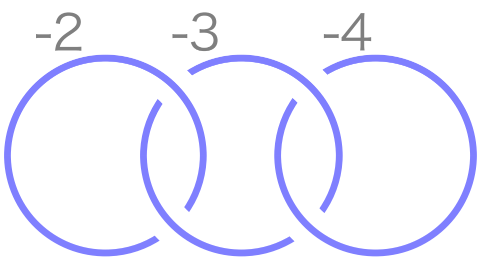
Note: This (with a suitable sign) is blow-up/down, related to resolution of the singularity.
The green curve is the exceptional curve.

It is not easy to find/prove “unexpected” lens space surgery!



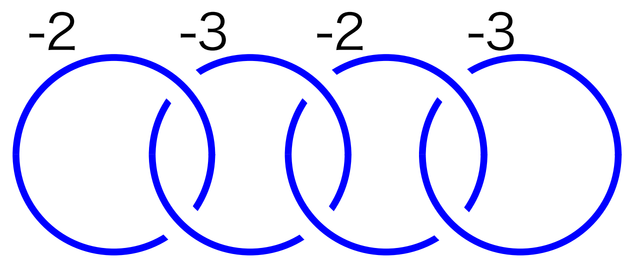
P(-2,3,7)

18-surgery is



?

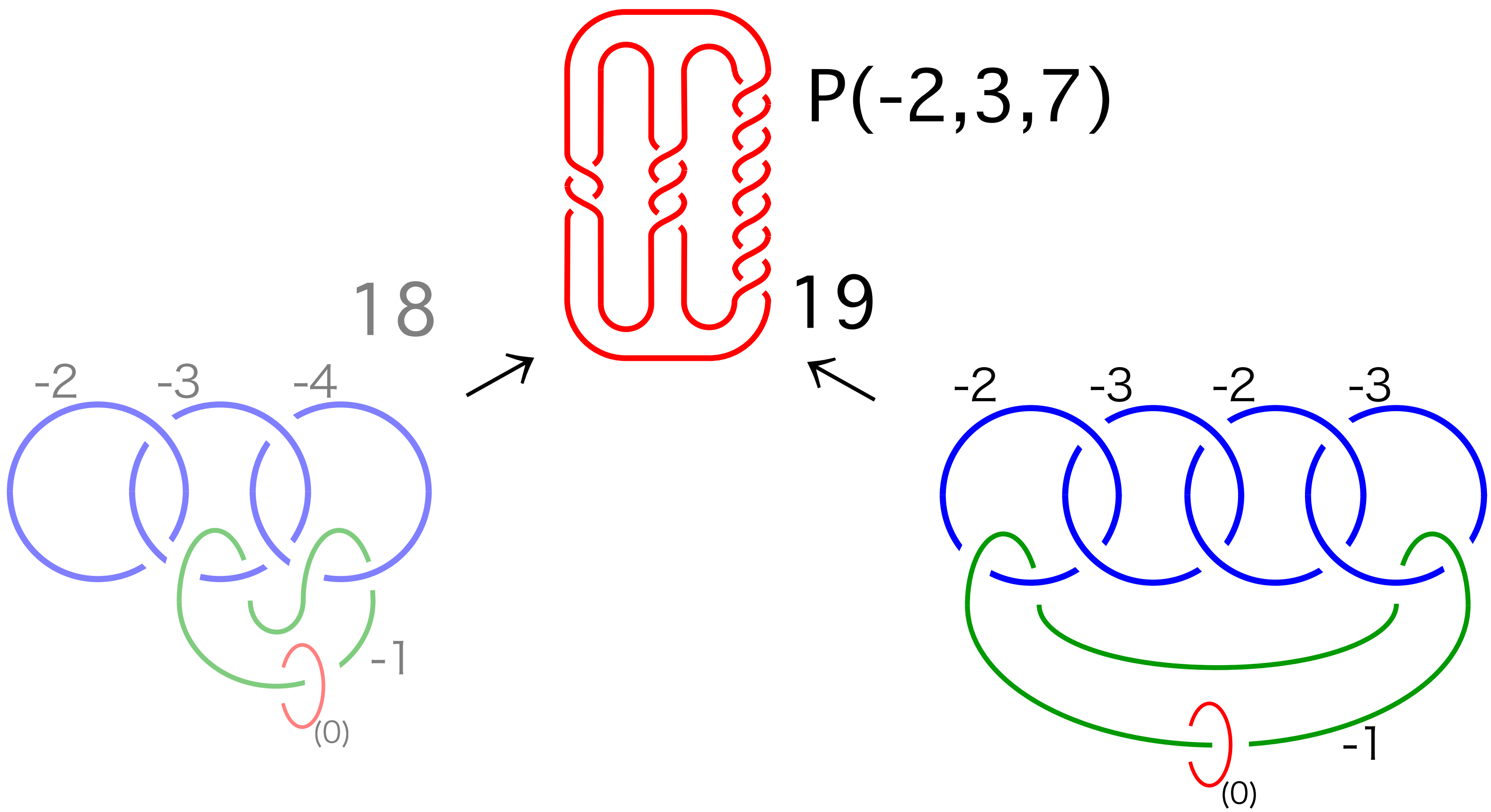
19-surgery is



Start of my research What is *the best method* to prove it?

One answer ([Y]) :

blue $\cup$ green $= S^3$, and red becomes the knot.

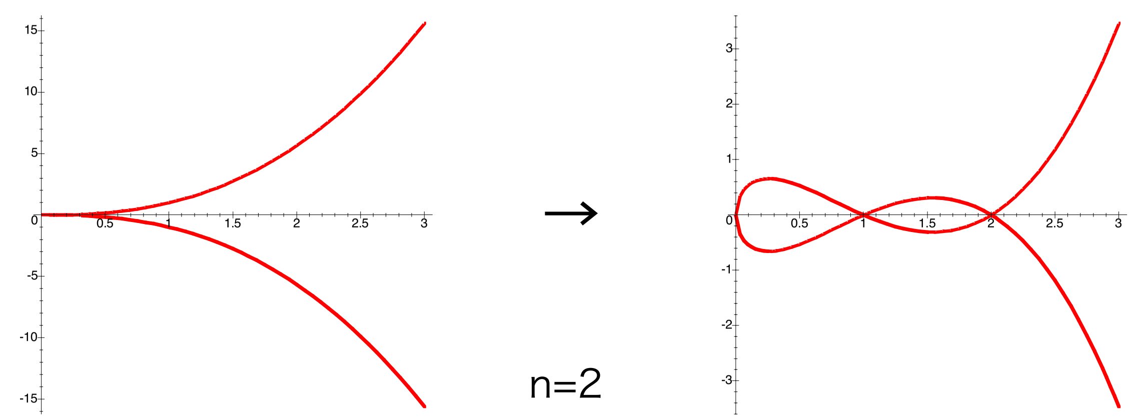


Only blow-downs (4 times)! ($\Rightarrow$ Resolution of singularity).

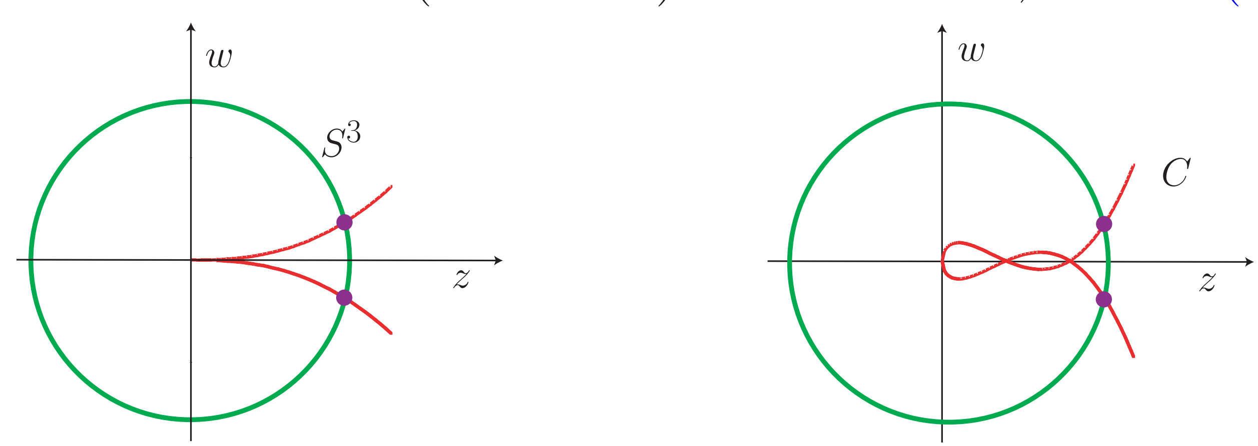
§2. A'Campo's divide knots, came from singularity theory.

In $\mathbf{R}^2$, purterb $y^2 = x^{2n+1}$ ("A_{2n}-sing.") and Draw the plane curve

$$C : \quad y^2 = x(x-\epsilon)^2(x-2\epsilon)^2 \cdots (x-n\epsilon)^2$$



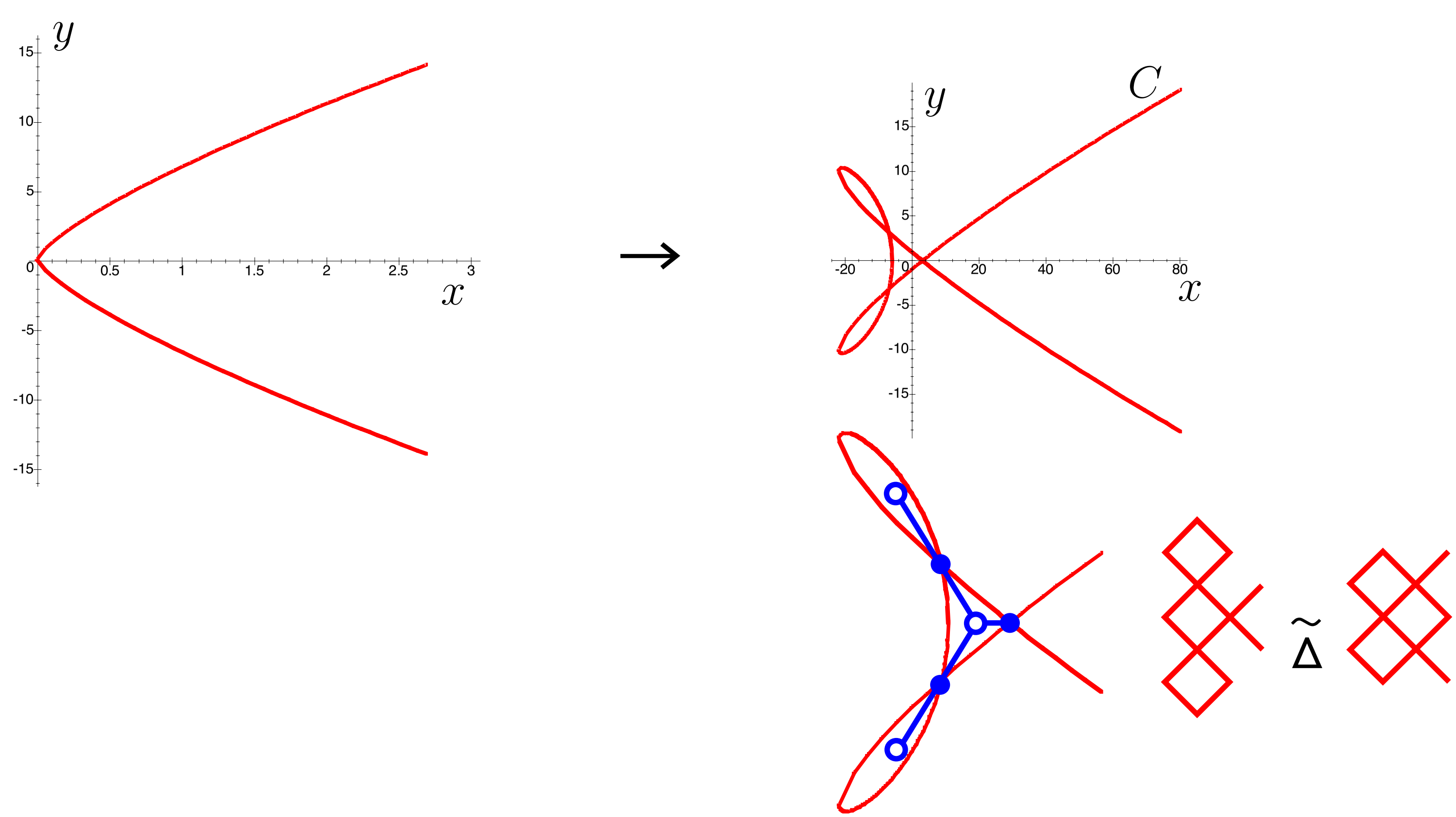
In $\mathbf{C}^2$, $C \cap S^3$ is a knot (or a link). For small ϵ, it is $T(2, 2n+1)$.



Another example [T.Urabe]

In $\mathbf{R}^2$, purterb $y^4 = x^3$ ("E_6-sing.") and Draw the plane curve

$$C : \quad (y^2 + \epsilon(6x + 32\epsilon^2))^2 - (x + 7\epsilon^2)^2(x + 22\epsilon^2) = 0$$



The knot is $T(4,3)$. We can see E_6 *Dynkin diagram.*

A'Campo generalized the correspondence

a generic plane Curve P $\Rightarrow$ a Link $L(P)$ in S^3

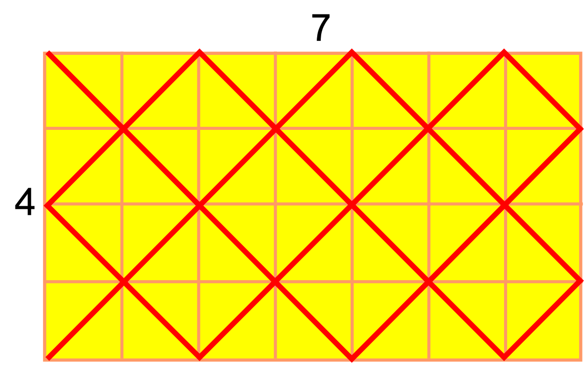
A'Campo's divide knots

Ex. Torus links [Goda-Hirasawa-Y, (Gusein-Zade, etc.)]

Rectangle curve $p \times q \;\Rightarrow\; T(p, q)$

(PL line with slope ± 1 in the rectangle.)

ex. $(p, q) = (7, 4)$



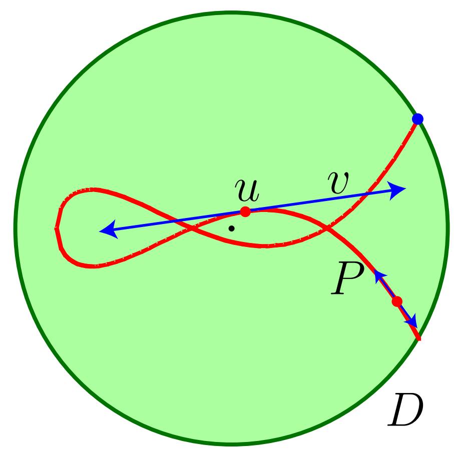
It has $\dfrac{(p-1)(q-1)}{2}$ double points, in general.

A'Campo's *original* construction.

a generic plane Curve P $\Rightarrow$ a Link $L(P)$ in S^3

A'Campo's divide knots

For "generic" (No self-tangency) proper curve P in the unit disk D,

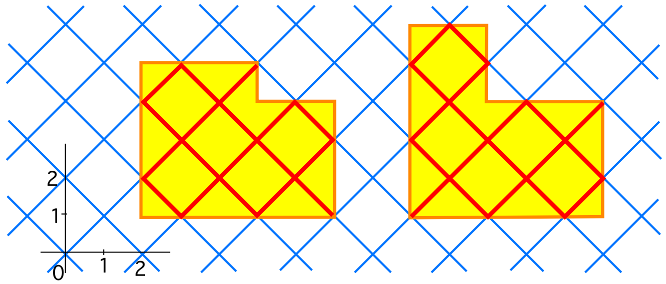


$$S^3 = \{(u, v) \in TD | u \in D,\ v \in T_u D,\ |u|^2 + |v|^2 = 1\}$$
$$L(P) := \{(u, v) \in TD | u \in P,\ v \in T_u P,\ |u|^2 + |v|^2 = 1\} \subset S^3.$$

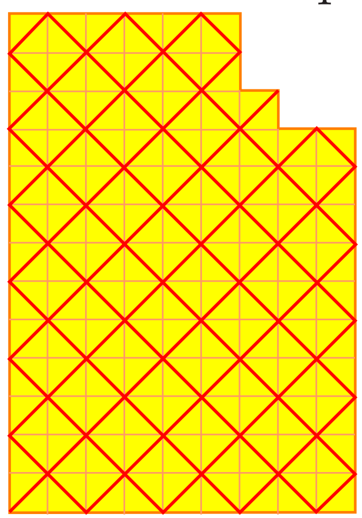
We use its visualization by [Hirasawa].

Classification of curves

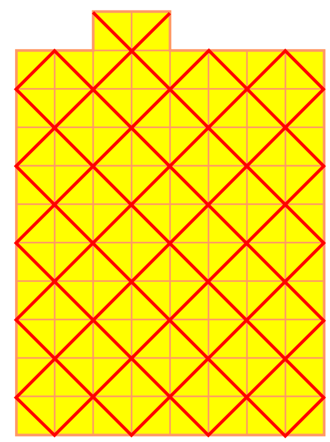
(1) L-shaped curve:



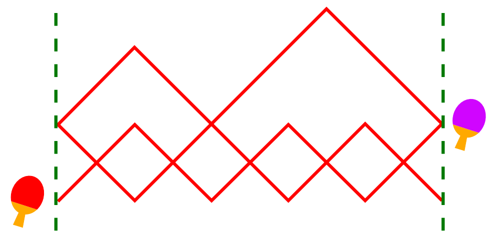
(2) generalized L-sheped curve,



(3) "凸" type curve

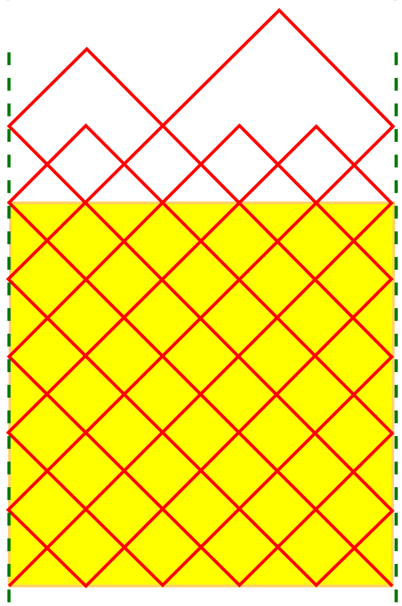


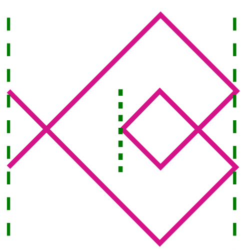
Assume: edges are vertical or horizontal, vertices are in $\mathbf{Z}^2 \subset \mathbf{R}^2$.



(4) “Pingpong type”

if (w.r.t at least one direction) max/min points are in the same level, up to isotopy.



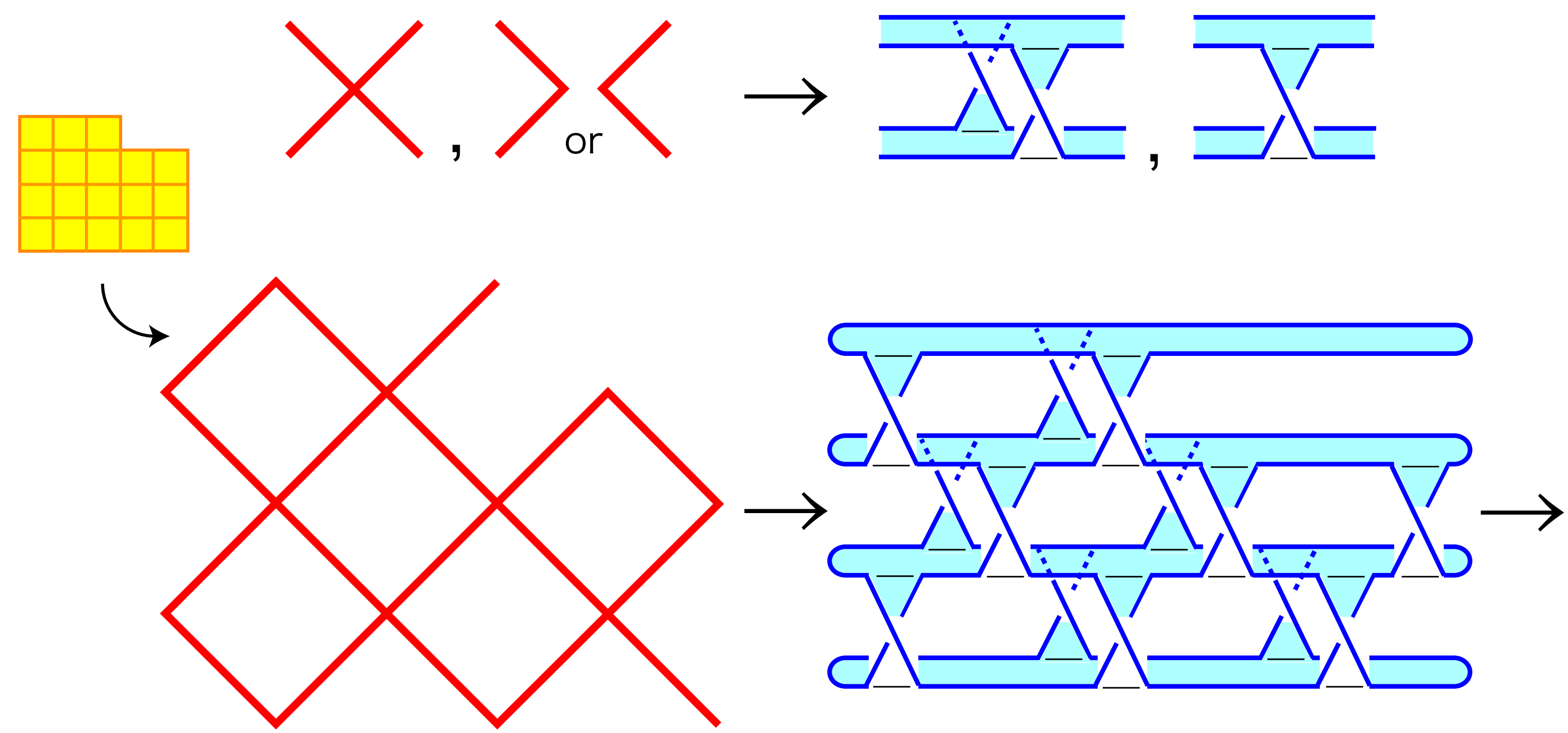


⇐ This is an example of Non-pingpong curve. (w.r.t horizontal nor vertical direction)

generalized L-shaped, 凸-type curves are pingpong type.

Curve P ⇒ Knot L(P). We get the braid presentation.

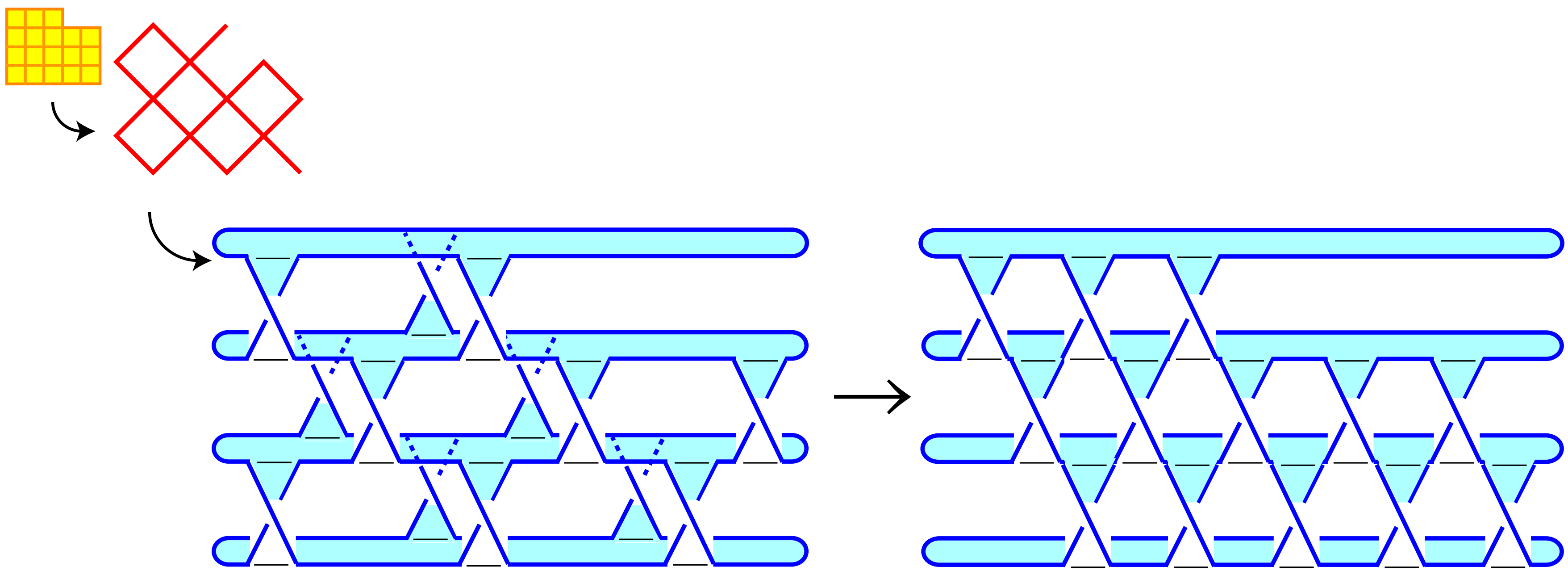
[Couture-Perron] only for pingpong curves ([Hirasawa] extends it)



$13 \cdot 21321 \cdot 2 \cdot 12132$

The closure is $P(-2, 3, 7)$, genus $= 5$.

and the fiber surface.



$$13 \cdot 21321 \cdot 2 \cdot 12132 \quad \sim \quad 321\ 321\ 321\ 21\ 21$$

We define a braid

$$W_n := \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1, \qquad \text{“}1/n \text{ twist”}$$

$P(-2,3,7)$ is presented by ${W_4}^3{W_3}^2$.

Basics on Divide knots [N.A'Campo, L.Rudolph,..]

(0) $L(P)$ is a knot ($\sharp L(P) = 1$) $\Leftrightarrow$ P is an immersed arc.

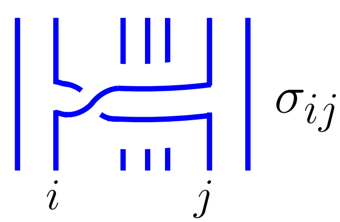
(1) The genus of knot $L(P)$ = $\sharp$ double points of P.

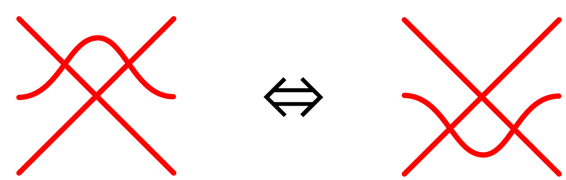
(2) $lk(L(P_1), L(P_2)) = \sharp(P_1 \cap P_2)$.

(3) Every divide knot $L(P)$ is *fibered*.

(4) Any divide knot is a closure of *strongly quasi-positive* braid.
i.e., product of some σ_{ij}.

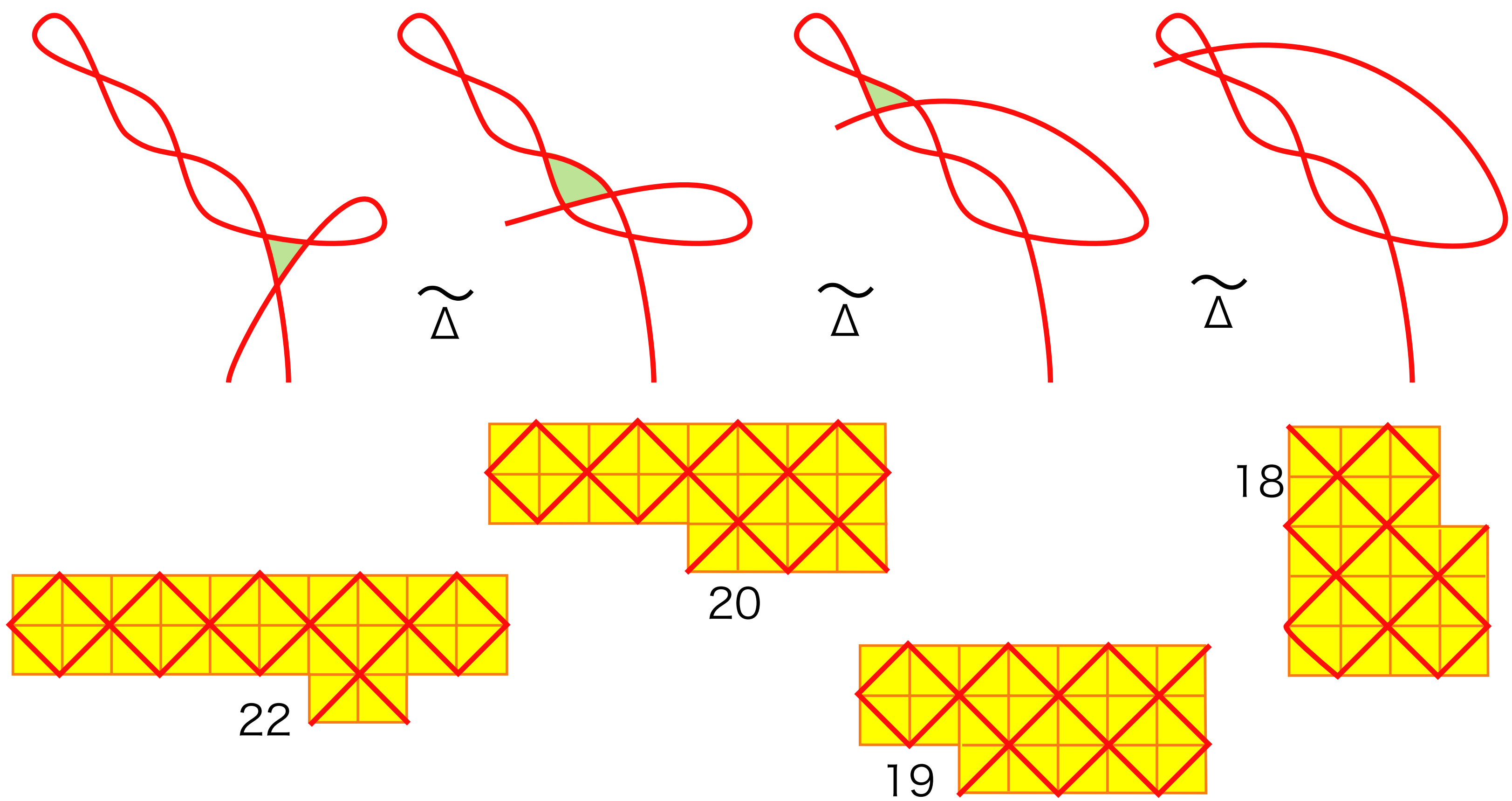
(5) $P_1 \sim P_2$ by Δ-move $\Rightarrow$ $L(P_1) = L(P_2)$. L is *not injective*.





Δ-move on plane curves

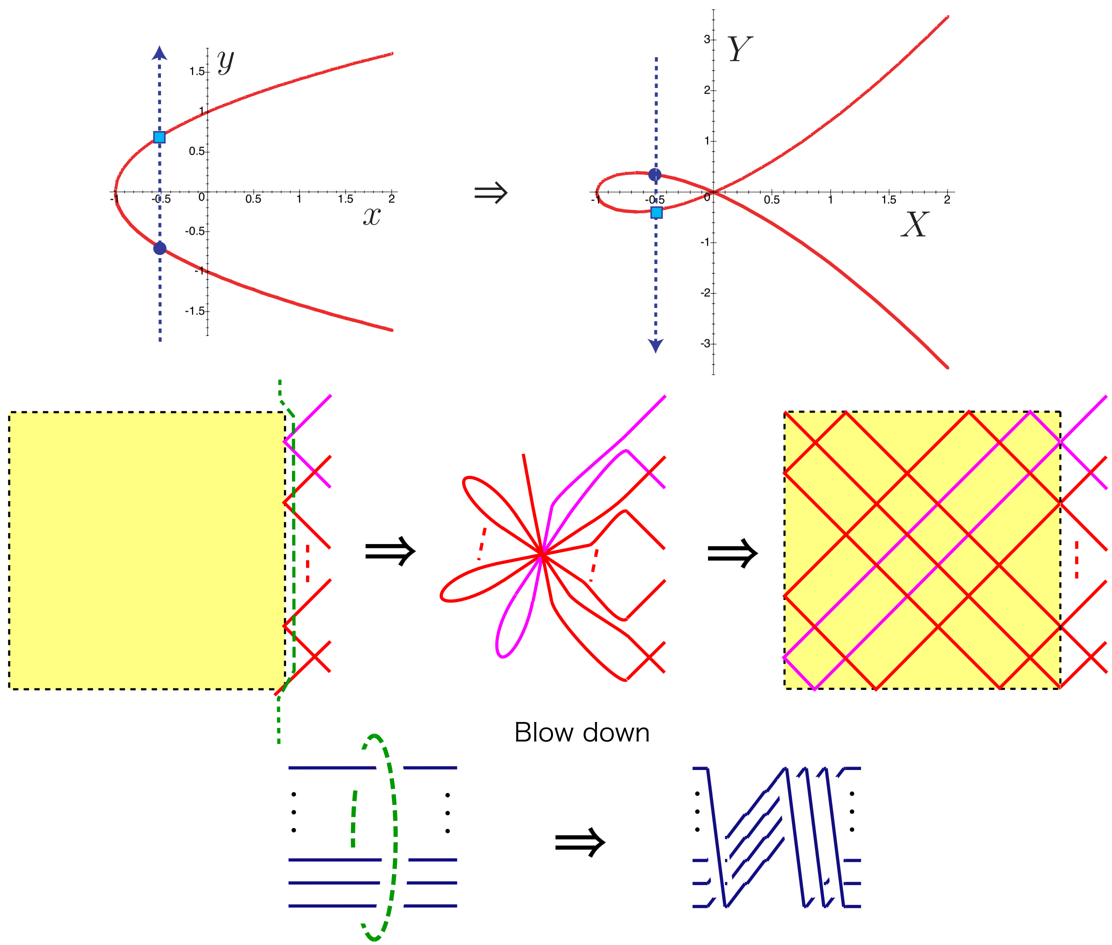
These curves present the same knot $P(-2,3,7)$
(Thanks to Hirasawa)



Lemma. [Y]

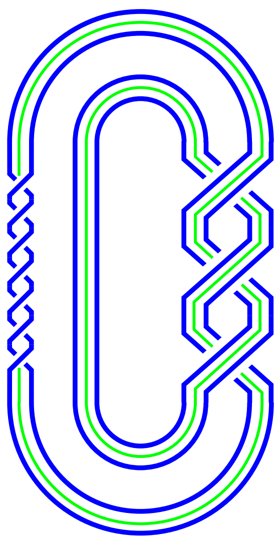
"**Adding a square**" corresponds to a right-handed **full-twist** = **blow-down** = coord. transform: $(x, y) = (X, Y/X)$.

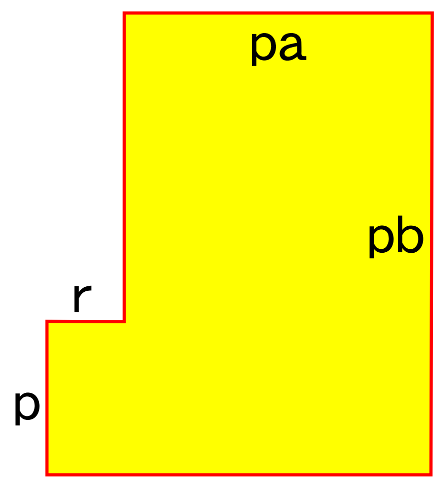
(ex. $y^2 = x + \epsilon$ becomes $Y^2 = X^2(X + \epsilon)$)

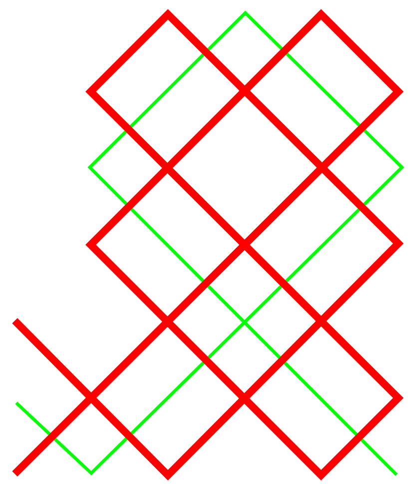


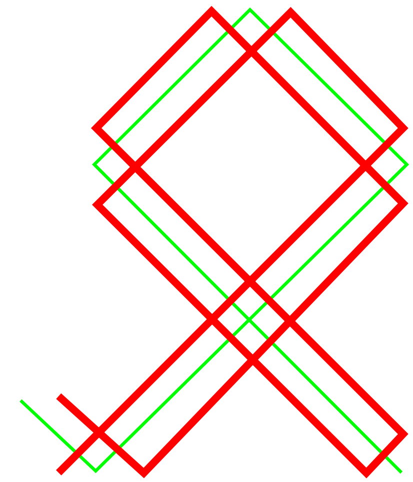
Cable knots of torus knots

$C(T(a,b);p,pab+r)$ is represented, (ex. $C(T(2,3);2,13)$)









It is algebraic: From $T(a,b) = \begin{cases} x = t^a \\ y = t^b \end{cases}$ to $\begin{cases} x = t^{ap} \\ y = t^{bp} + t^{bp+r} \end{cases}$, or

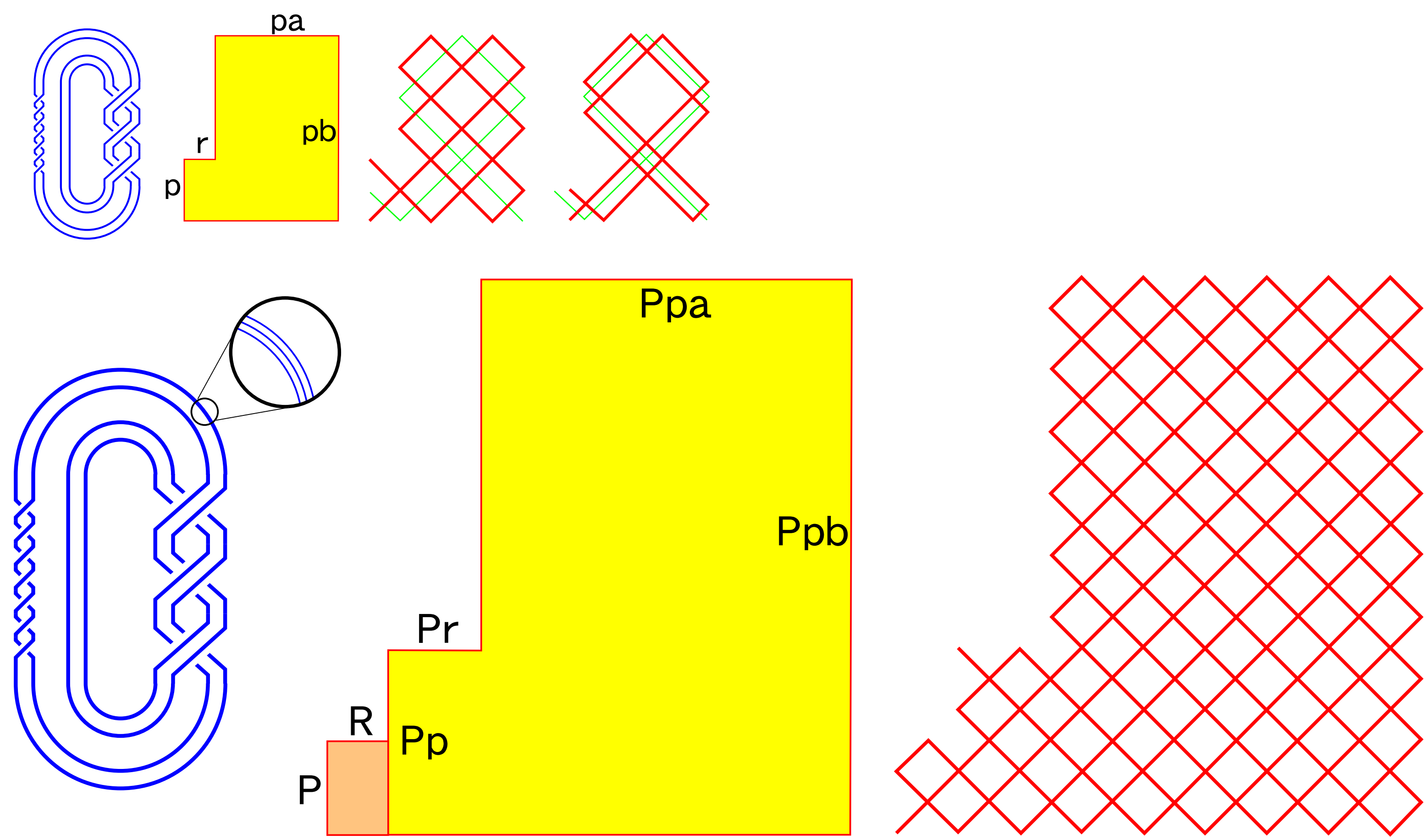
$$y = x^{\frac{b}{a}}\left(1 + x^{\frac{r}{ap}}\right)$$

Puiseux pair is $\{(b,a),(bp+r,p)\}$.

More iterated cables ($\Rightarrow$ generalized L-shaped)

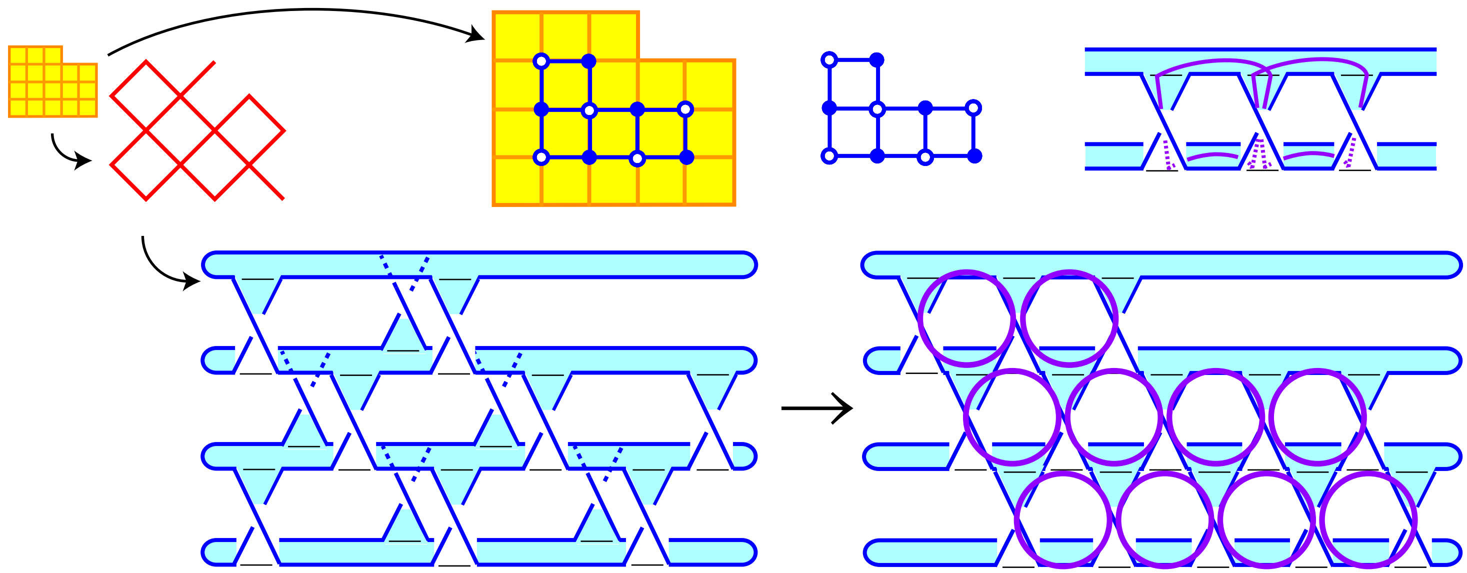
$C(\,C(T(a,b);p,pab+r);P,Pp(pab+r)+R))$ is represented,

(ex. $C(\,C(T(2,3);2,13);3,80))$, $\;y = x^{\frac{b}{a}}\left(1 + x^{\frac{r}{ap}}\left(1 + x^{\frac{R}{apP}}\right)\right)$.



Lemma. Decomposition of monodromy

For an (generalized) L-shaped divide P, The monodromy φ of the fiber surface of $L(P)$ can be decomposed as a product of some positive Dehn twists along the "grapes" of P.



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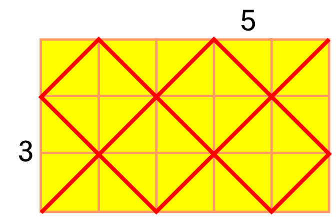
Middle disks are Murasugi-disks. Each floor is obtained by plumbings. Thus the monodromy is the product (from the bottom and the left).

§3. Lens space surgery “Which $(K;p)$ is a lens space?”

ex.1 [’71 L. Moser] **Torus knots.**

$$p = ab \pm 1 \;\Rightarrow\; (T(a,b);p) \cong L(p,-b^2).$$

$K := T(3,5)$, then $(K;16) = L(16,7)$ and $(K;14) = L(14,5)$.



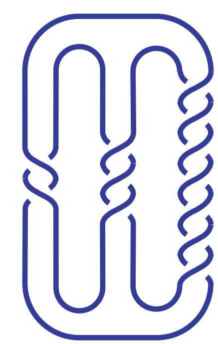
ex.2 [’77 J. Bailey, D. Rolfsen] **2 Cables of Torus knots**

— Shown in §2. —

ex.3 [’80 R. Fintushel, R. Stern] Hyperbolic knot!

$K := P(-2,3,7)$, then $(K;19) = -L(19,7)$.

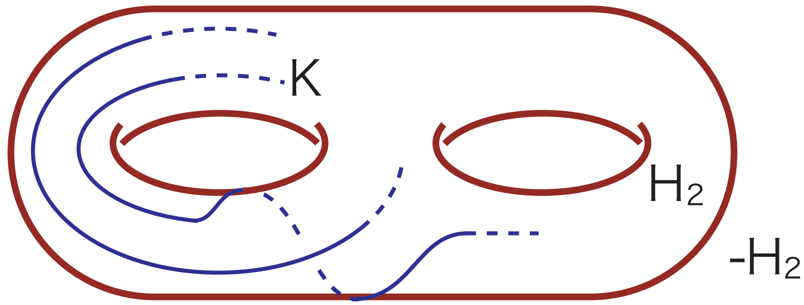
$(K;18) = -L(18,7)$.



Berge's doubly-primitive knots ['90]

A knot K in the Heegaard surface Σ_2 is *doubly-primitive* iff

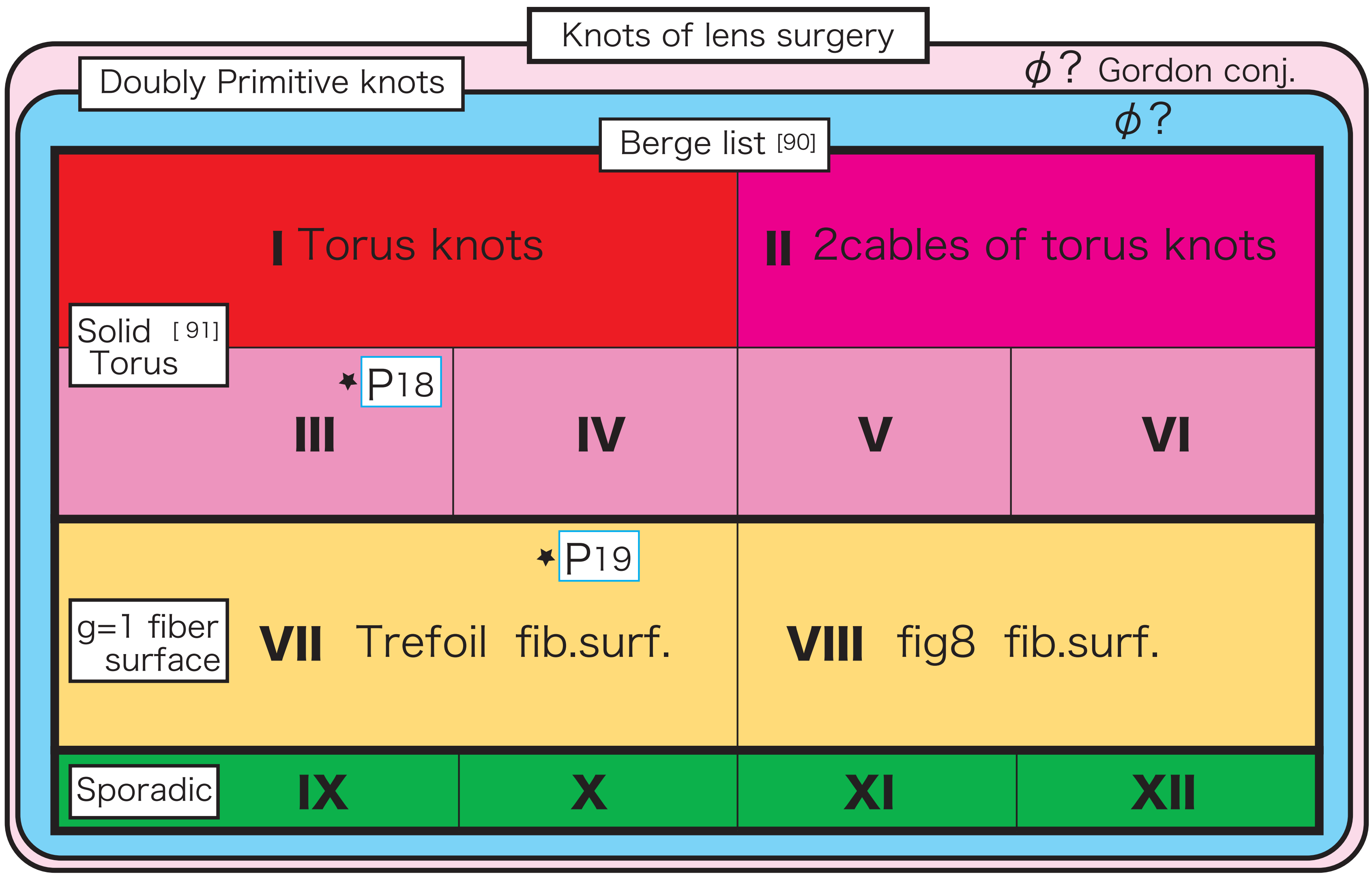
$$K_\sharp \text{ (as in } \pi_1\text{) is a generator in both } \pi_1(H_2) \text{ and } \pi_1(-H_2).$$



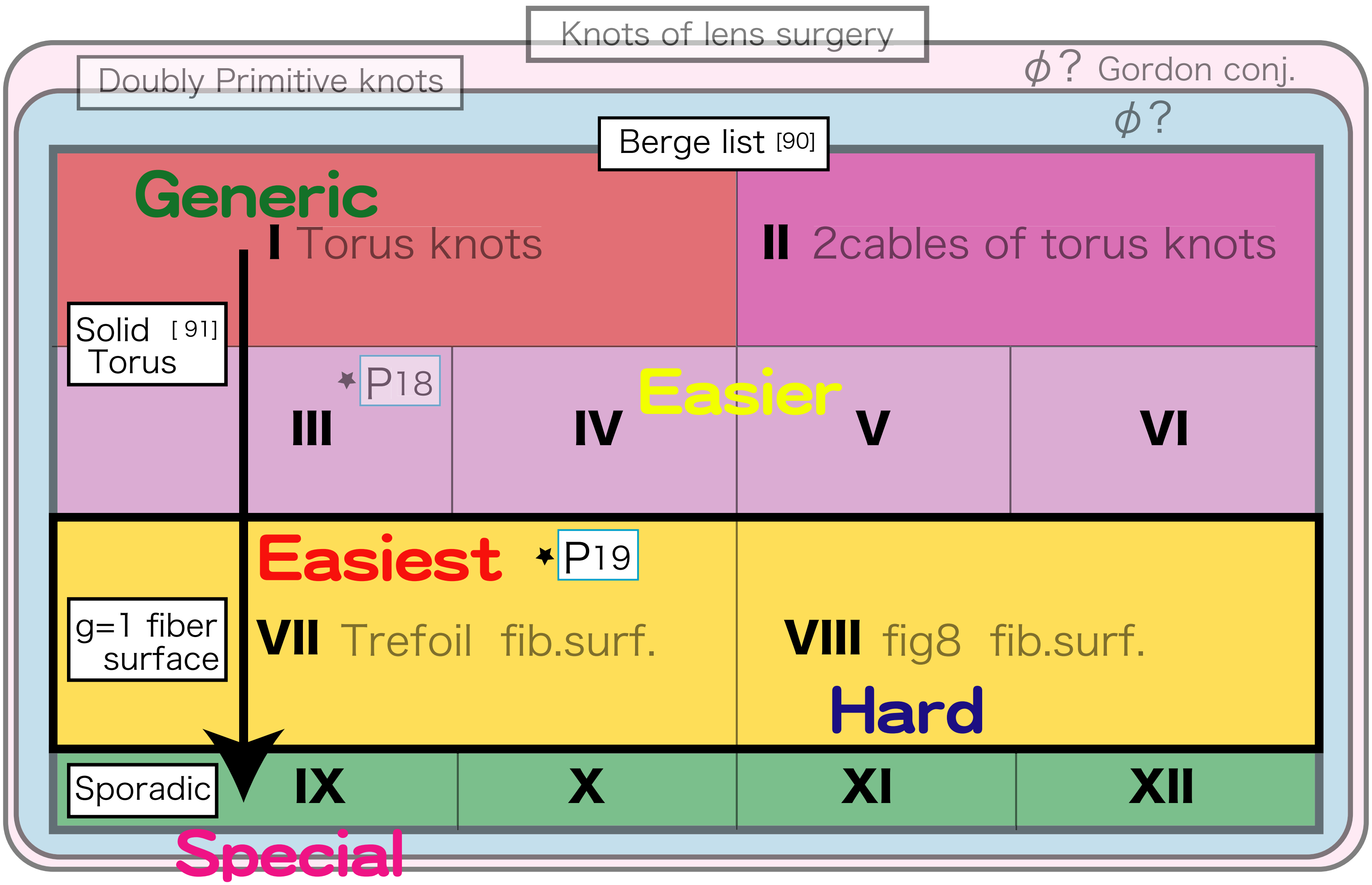
Such a knot K with the surface slope (coeff.) always yields a lens space. ■

Berge (tried to) classfied and made a list of such knots.
His list consists of **3** Families, and of **12** "Type"s.

Type I, II, III, $\cdots$, VI | VII, **VIII** | IX, $\cdots$, XII.



Knots of lens surgery
Doubly Primitive knots
ϕ? Gordon conj.
ϕ?
Berge list [90]
I Torus knots
II 2cables of torus knots
Solid [91] Torus
P18
III
IV
V
VI
P19
g=1 fiber surface
VII Trefoil fib.surf.
VIII fig8 fib.surf.
Sporadic
IX
X
XI
XII


Knots of lens surgery
Doubly Primitive knots
ϕ? Gordon conj.
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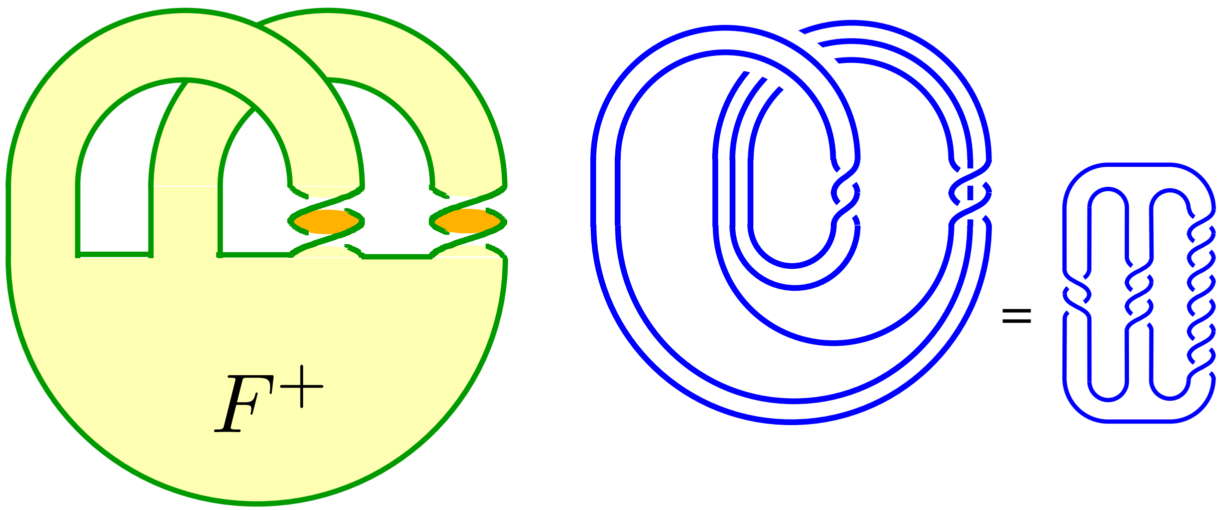
Berge list [90]
Generic
I Torus knots
II 2cables of torus knots
Solid [91]
Torus
P18
III
IV
Easier
V
VI
Easiest
P19
g=1 fiber
surface
VII Trefoil fib.surf.
VIII fig8 fib.surf.
Hard
Sporadic
IX
X
XI
XII
Special

TypeVII ([Berge], see also [Y]) Let (a, b) be coprime (positive).
Let F^+ be the fiber surface of the left-handed trefoil.
A knot $k^+(a, b)$ is defined as below: p-surgery is $L(p, q)$.
($p = a^2 + ab + b^2$, $q = -(a/b)^2 \bmod p$)



- $k^+(2, 3)$ is $P(-2, 3, 7)$. $\quad 2^2 + 2 \cdot 3 + 3^2 = 19$.

$k^+(a, b)$ is obtained from $T(a, b)$ by $+$ full-twist twice.

§4. **Results** (old and new)

Theorem A. (L-shaped) ([Y'06–'07])
Every knot (up to mirror image) in Type**I**, **II**, **III** $\cdots$ **VI**,

- is a *divide knot*,
- is presented by an *L-shaped plane curve* s.t.

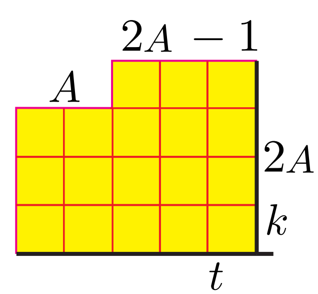
$$\mathrm{Area}(L) - \mathrm{coeff.} \;=\; 0 \text{ or } 1.$$

Let (a, b) (positive) coprime.
Every knot $k(a, b)$ in **VII** is presented by an *L-shaped plane curve* s.t. $\mathrm{Area}(L) = \mathrm{coeff.}$. ■

Families of lens surgery including $K := P(-2,3,7)$

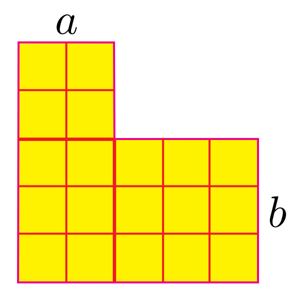
$(K,18)$ and $(K,19)$ belongs to different family.

Type**III** $(\ni (K;18))$



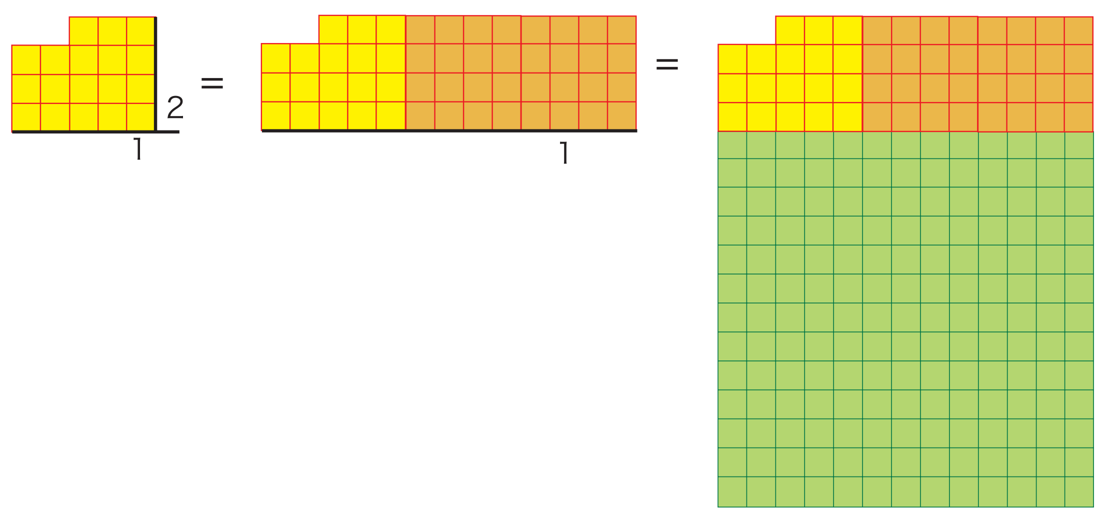
$(A=2, k=t=0)$

Type**VII** $(\ni (K;19))$



$((a,b)=(2,3))$

Notation:

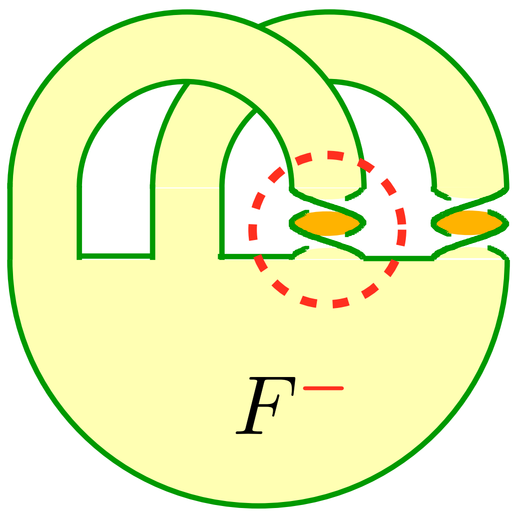


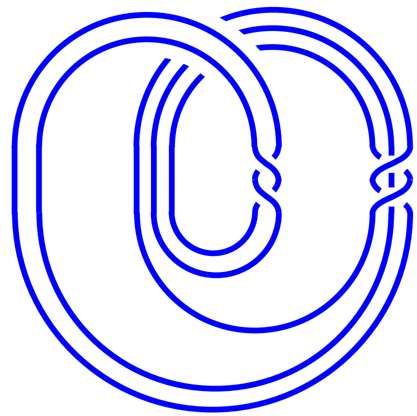
TypeVIII ([Berge]) – The most difficult Type –

Let F^{-} be the fiber surface of Fig8 knot.

A knot $k^{-}(a,b)$ is defined as below: p-surgery is $L(p,q)$.

($p = -a^2 + ab + b^2$, $q = -(a/b)^2 \bmod p$)





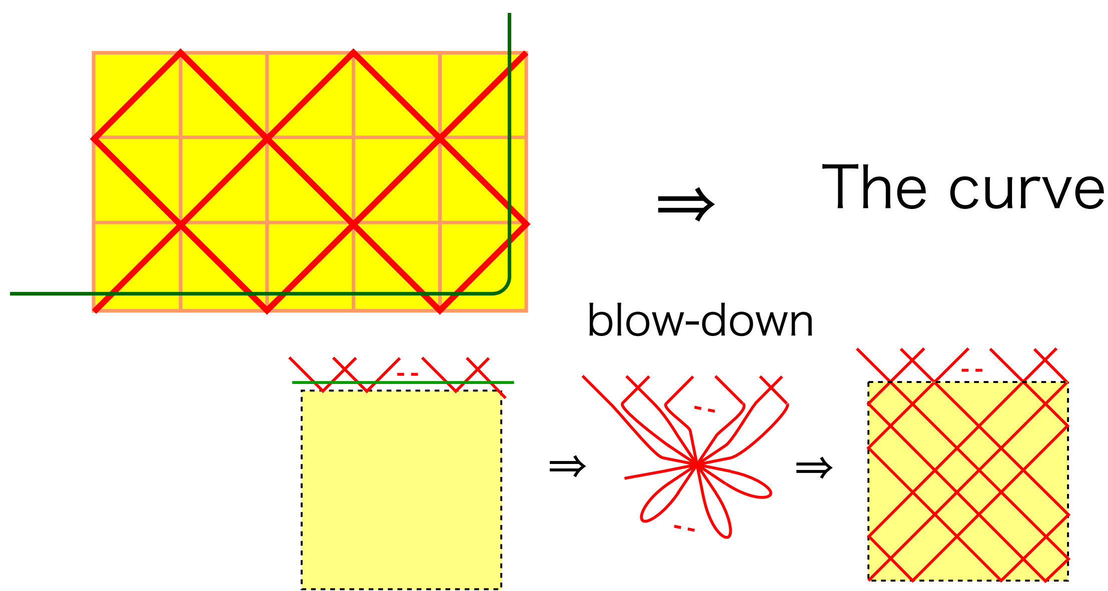
- $k^{-}(2,3)$ is $T(3,4)$ (unfortunately). $-2^2 + 2 \cdot 3 + 3^2 = 11$.

$k^{-}(a,b)$ is obtained from $T(a,b)$ by full-twist twice, $+$ and $-$.

Note that $k^{-}(a,b) = k^{-}(b-a,b)$.

Main Theorem ([Y'08]) Let (a, b) coprime (and $0 < a < b$ here). Every **TypeVIII** knot $k^-(a, b)$ is a *divide knot*. The plane curve is obtained by a *blow-down* from the rectangle curve $a \times (b - a)$, as follows:

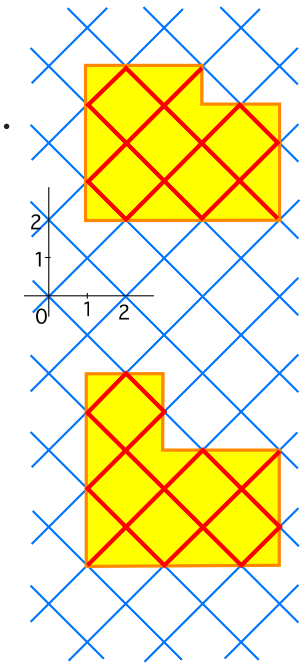
ex. $k^-(3, 8)$



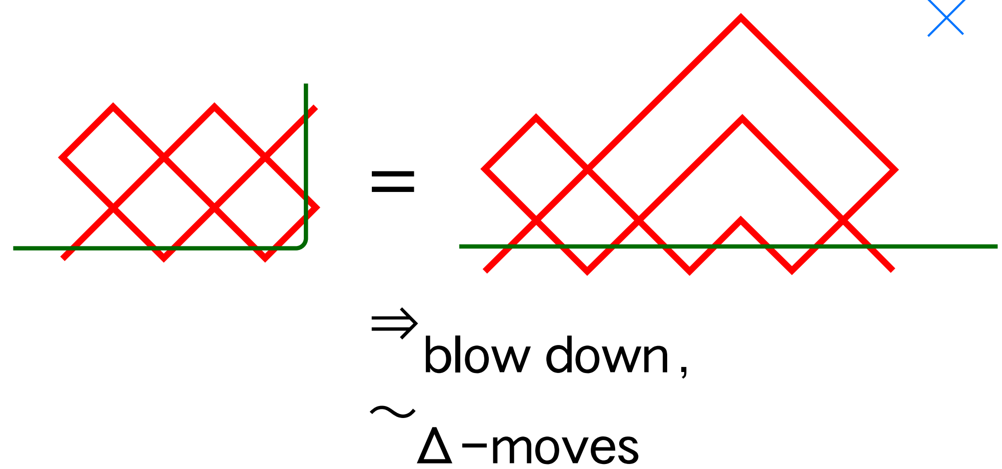
■

Next Question: Which (type) is the curve?

I want to deform the curves as good as possible.
I hoped it is L-shaped curve.

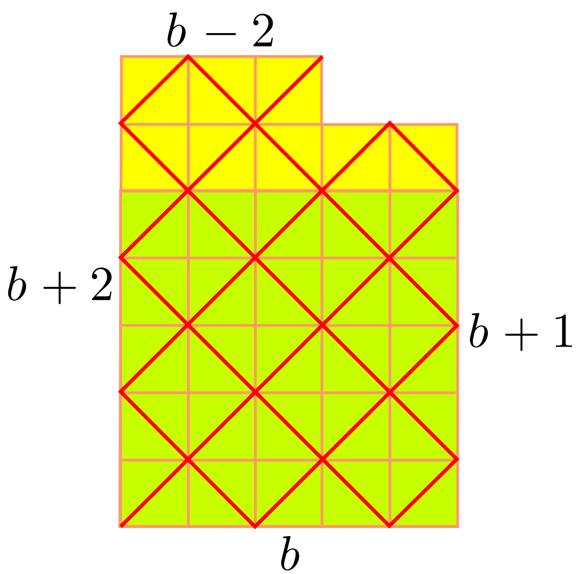


We can use Δ-moves.

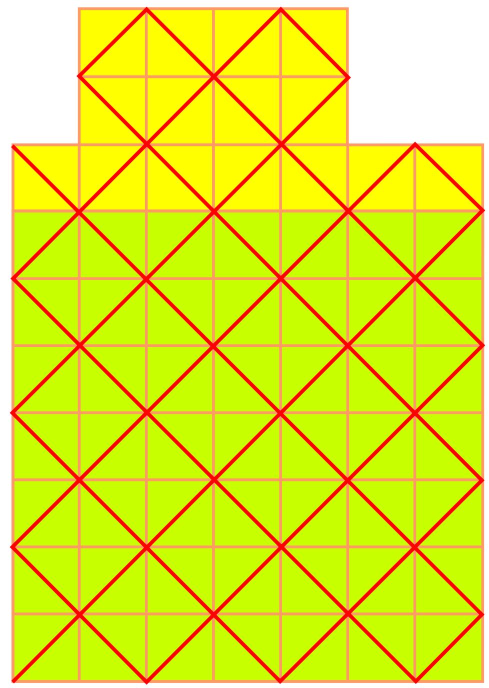


The curve is, at least, pingpong type.

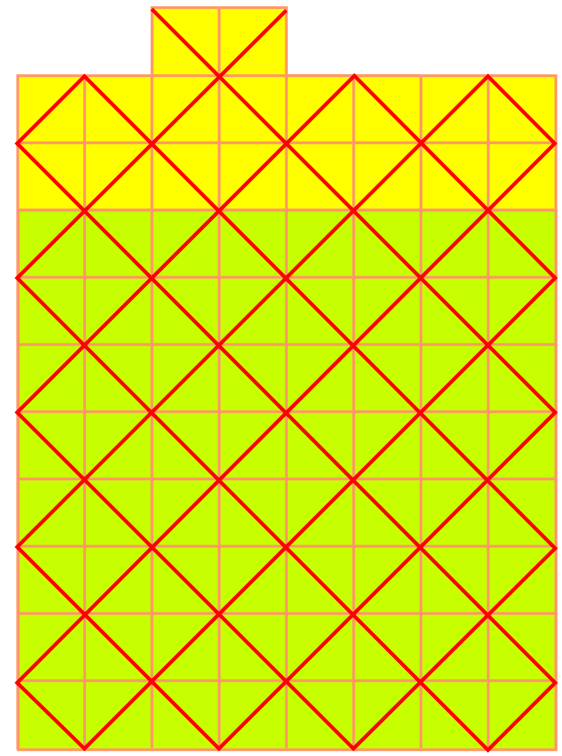
Trial(up to now):



$k^-(2,b)$



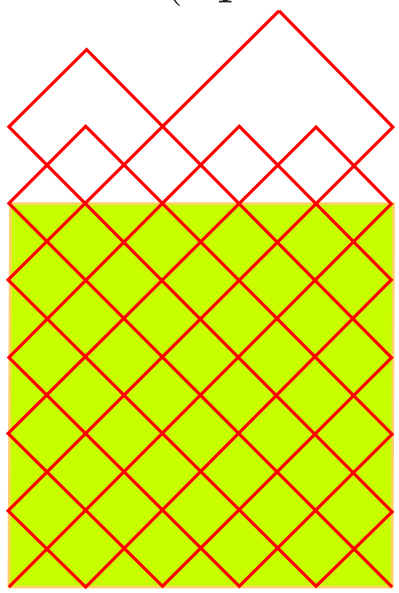
$k^-(3,7)$



$k^-(3,8)$

	$k^-(2,b)$	$k^-(3,7)$	$k^-(3,8)$
Area	b^2+2b-2	64	82
Coeff.	b^2+2b-4	61	79

Trial(up to now):

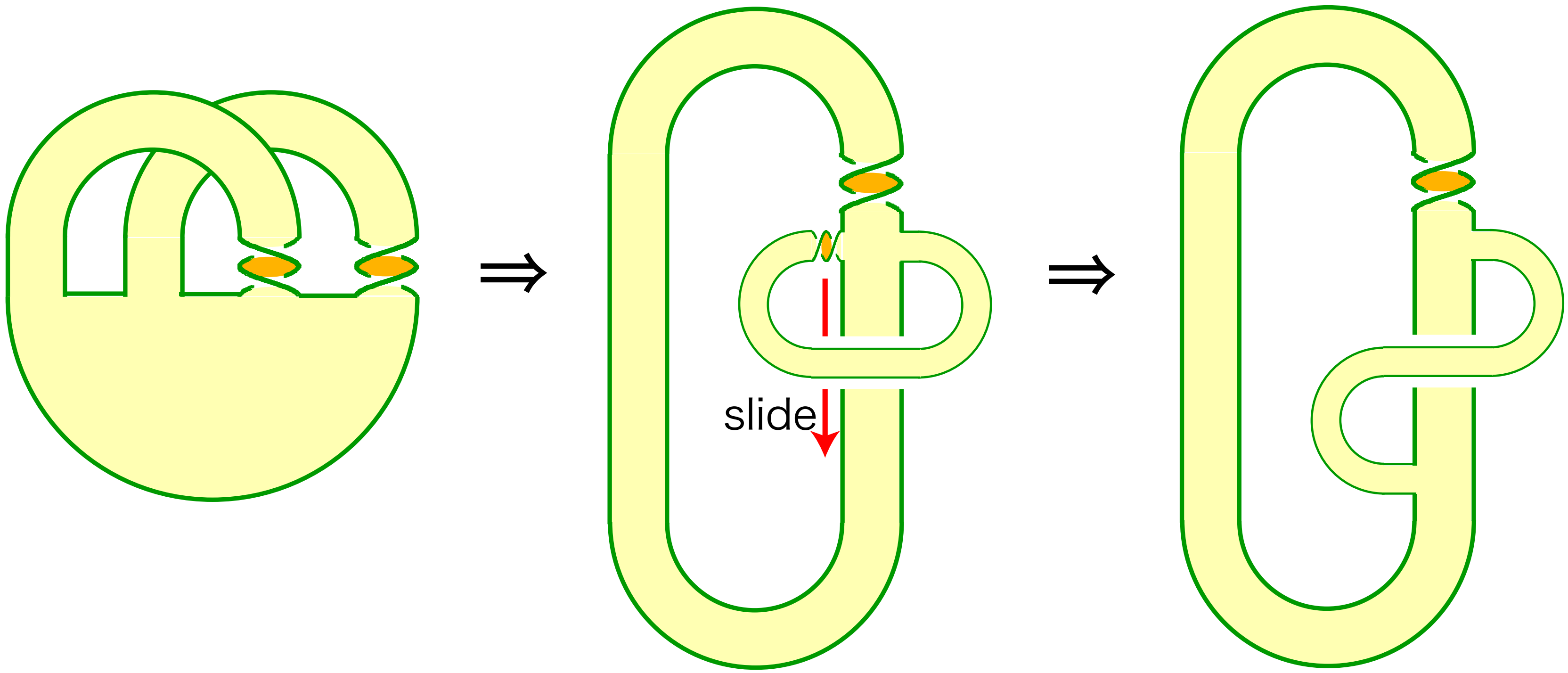


$k^-(3, 10)$

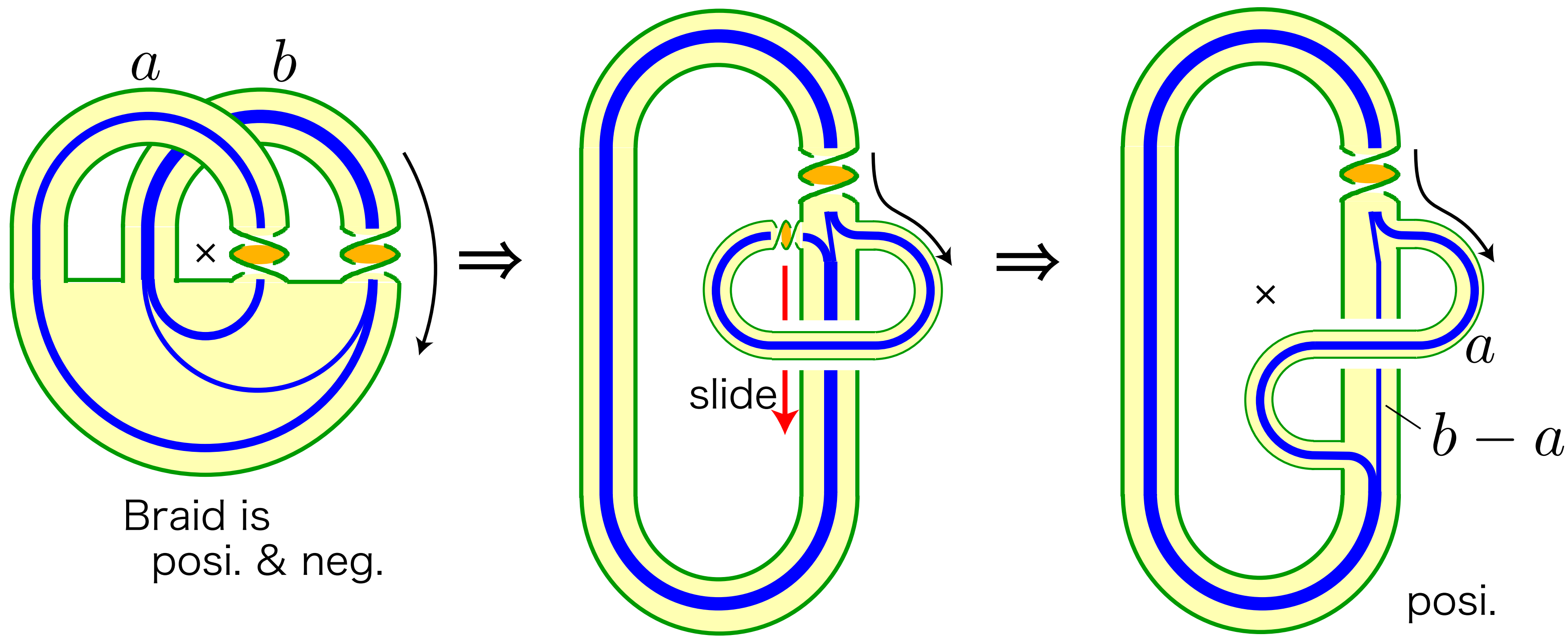
The curves (for TypeVIII) are at least pingpong type.

Proof of Main Thm.

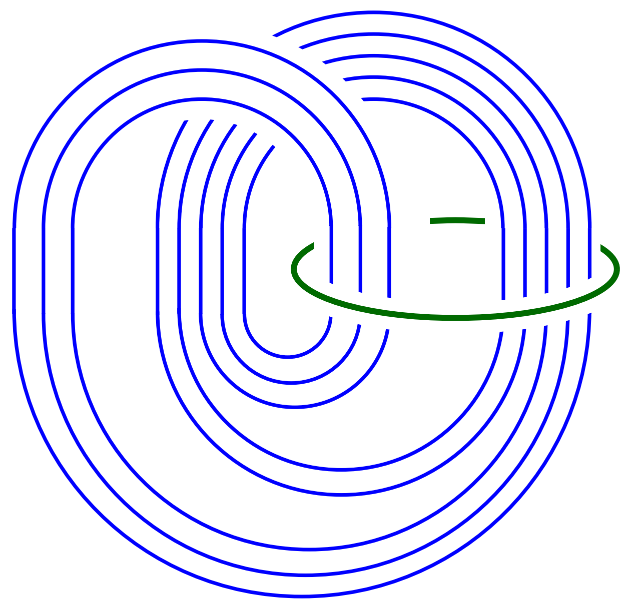
Use Baker's deformation of the surface F^{-1}



The curve $k^{-}(a,b)$ in F^{-1} becomes a positive braid of index b



It shows that $k^{-}(a,b)$ is obtained by $+1$ full-twist from $T(a,b-a)$



ex. $k^{-}(3, 8)$ is from $T(3, 5)$

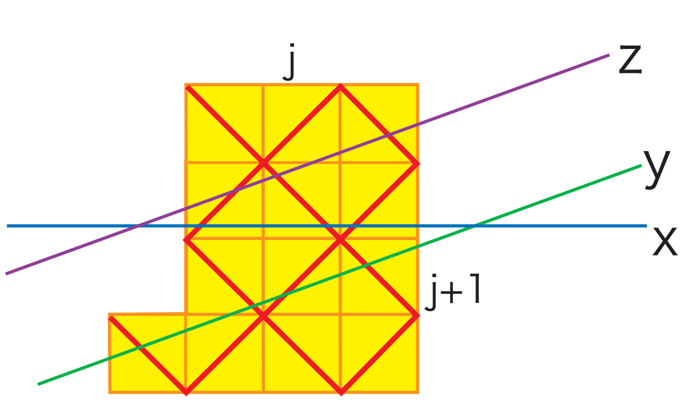
Find the curve presenting this braid and the axis. □

Remark: 4-dim. Topology ([Y])
This diagram is J.Park's rational homology 4-ball ($B_{p,q}$), used in the *generalized rational blow-down*, whose boundary is a lens space $L(p^2, pq - 1)$.

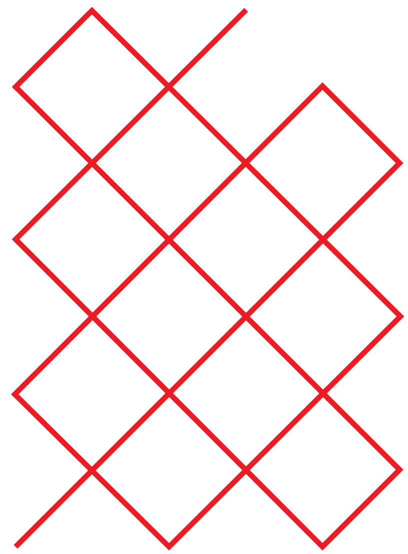
Trial to Spradic examples (the most special family)

Type**IX**: blow-down in order x, y, z $\quad (p = 22j^2 + 9j + 1)$

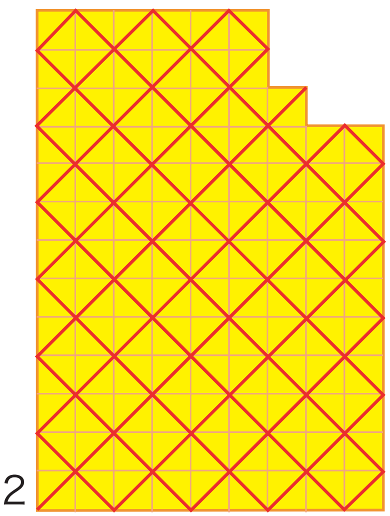
Type **X**: blow-down in order x, z, y $\quad (p = 22j^2 + 13j + 2)$



ex.1



ex.2



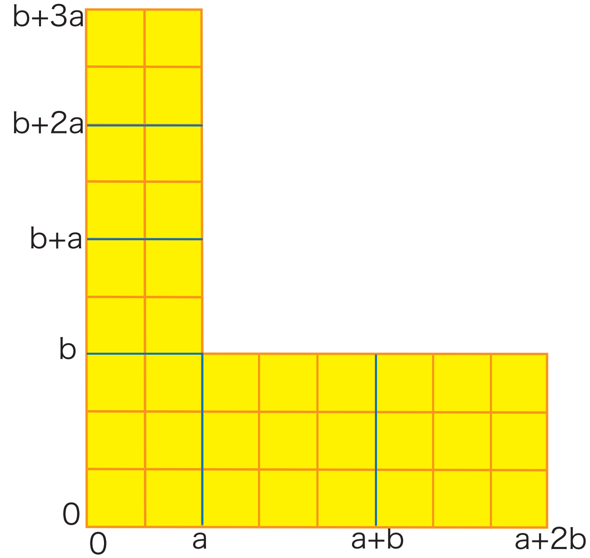
ex.1 (**IX** $j = 1$) L-shaped, $p = 32$, Area=33.

ex.2 (**IX** $j = 2$) generalized L-shaped, $p = 107$, Area=109.

By the way [Y'05]

$\exists$ L-shaped curve that represents a knot whose

- Area-surgery is not lens,
- does not admit a lens space surgery.



This family contains $P(-2, 3, 2n+5)$ with $n > 2$ [Bleiler-Hodgson], which are known *not* to admit lens space surgeries, but Seifert surgery.

(L-shaped) Curves still tends to present interesting Dehn surgery.

Thank you very much!